Week 4 (disc 105)

Wednesday, February 10, 2021 10:49 AM

Math 128a – Week 4 Worksheet
GSI: Izak, (2/10/21)

2.3 Problems

Problem 1. *Come up with a function $f \in C^2[a,b]$ with $f(p) = 0$ for some $p \in [a,b]$ such that Newtons method fails to converge for any initial guess not equal to p.*

Problem 2. *Derive the error formula for Newton's method:*

$$|p - p_{n+1}| \le \frac{M}{2|f'(p_n)|}|p - p_n|^2$$

2.4 Problems

Problem 3. *Generalize one of your homework problems. Construct a sequence p_n converging to p at order α with asymptotic error constant λ.*

2.5 Problems

Problem 4. *Steffensen's method is applied to a function $g(x)$ using $p_0^{(0)} = 1, p_2^{(0)} = 3$ to obtain $p_0^{(1)} = .75$. What is $p_1^{(0)}$?*

Problem 5. *Prove that if p_n converges linearly to p and $\lim_{n\to\infty}\frac{p_{n+1}-p}{p_n-p} < 1$, then $\lim_{n\to\infty}\frac{\hat{p}_n-p}{p_n-p} = 0$ where $\hat{p}_n$ comes from Aitken's Δ^2 method. (Hint: let $\delta_n = (p_{n+1}-p)/(p_n-p) - \lambda$ and show that $\lim_{n\to\infty}\delta_n = 0$. Then express $(\hat{p}_{n+1}-p)/(p_n-p)$ in terms of δ_n, δ_{n+1} and λ).*

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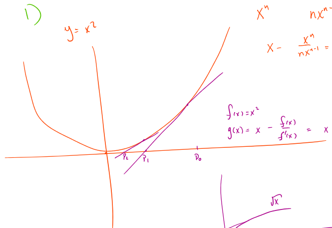
1)

$y = x^2$

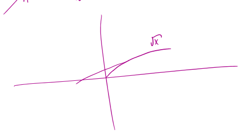


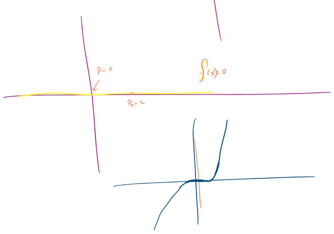
$x^n \quad nx^{n-1}$

$x - \frac{x^n}{nx^{n-1}} = x - \frac{x}{n} = \frac{(n-1)x}{n}$

$f(x) = x^2$

$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2}{2x} = x - \frac{x}{2} = \frac{x}{2}$





$|P - P_{n+1}| \le \frac{M}{2|f'(P_n)|}|P - P_n|^2 \qquad M = \max|f''(x)|$

$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$ Newton's method

$P_n - P_{n+1} = \boxed{\frac{f(P_n)}{f'(P_n)}}$

@ Taylor at $x = p$

$f(x) = \underbrace{f(p)}_{0} + f'(p)(x-p) + f''(\xi)\frac{(x-p)^2}{2}$

$= f'(p)(x-p) + f''(\xi)\frac{(x-p)^2}{2}$

$\frac{f(P_n)}{f'(P_n)} = \frac{f'(p)(P_n-p)}{f'(P_n)} + \frac{f''(\xi)(P_n-p)^2}{2f'(P_n)}$

$P - P_n + P_n - P_{n+1} = \frac{f'(p)(P_n-p)}{f'(P_n)} + \frac{f''(\xi)(P_n-p)^2}{2f'(P_n)} + P - P_n$

$P - P_{n+1} = \left(\frac{f'(p)}{f'(P_n)} - 1\right)(P_n - P) + \frac{f''(\xi)(P_n-p)^2}{2f'(P_n)}$

$f'(P_n) = f'(p) + f''(\xi)(P_n - p)$

$\frac{f'(P_n)}{f'(p)} = 1 + \frac{f''(\xi)(P_n-p)}{f'(p)}$

$= \frac{f'(p) + f''(\xi)(P_n-p)}{f'(p)}$

$\left(\frac{f'(p)}{f'(P_n)} - 1\right) = \frac{f'(p)}{f'(p) + f''(\xi)(P_n-p)} - 1 = \frac{\cancel{f'(p)} - \cancel{f'(p)} - f''(\xi)(P_n-p)}{f'(p) + f''(\xi)(P_n-p)}$

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instead let $g(x) = x - \frac{f(x)}{f'(x)}$ so $P_{n+1} = g(P_n)$

$g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}$

$g'(p) = 1 - \frac{(f'(p))^2}{(f'(p))^2} = 0$

$g(p) = p$

$$g'(p) = 1 - \frac{(f'(p))^2}{(f'(p))^2} = 0$$

$$g(p) = p$$

$$g(x) = p + \frac{g''(\xi)(x-p)^2}{2}$$

$$p_{n+1} - p = g(p_n) - p = \underbrace{p + \frac{g''(\xi)(p_n-p)^2}{2}}_{g(x)|_{x=p_n}} - p = \frac{g''(\xi)(p_n-p)^2}{2}$$

$$\left(\frac{f'(p)}{f'(p_n)} - 1\right) = \frac{f'(p)}{f'(p)+f''(\xi)(p_n-p)} - 1 = \frac{\cancel{f'(p)} - \cancel{f'(p)} - f''(\xi)(p_n-p)}{f'(p)+f''(\xi)(p_n-p)}$$

$$= \frac{-f''(\xi)(p_n-p)}{f'(p)+f''(\xi)(p_n-p)}$$

$$|p_{n+1} - p|$$

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

$$p_{n+1} - p_n = -\frac{f(p_n)}{f'(p_n)}$$

$$p_{n+1} - p_n + p_n - p$$

$$= \left| \frac{-(p_n-p)^2 f''(\xi)}{f'(p)+f''(\xi)(p_n-p)} + \frac{f''(\xi_2)(p_n-p)^2}{2f'(p_n)} \right|$$

$$\leq \frac{M\,|p_n-p|^2}{\underbrace{|f'(p)+f''(\xi)(p_n-p)|}} + \boxed{\frac{M\,|p_n-p|^2}{2|f'(p_n)|}}$$

$$|f'(p)+f''(\xi)(p_n-p)| \geq$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}$$

$$g''(x) = -\frac{(f'(x))^2\left(2f'(x)f''(x) - f'(x)f''(x) - f(x)f'''(x)\right) - 2f'(x)f''(x)\left((f')^2 - ff''\right)}{(f')^4}$$